

Inverse Z Transform

Methods for obtaining IZT

- ① Power series expansion
- ② Partial fraction expansion
- ③ Residue method.

Power Series Expansion Method $\Rightarrow$ is also called

long division method or direct division method

$$X(Z) = a_0 + a_1 Z^{-1} + a_2 Z^{-2} + \dots + a_n Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

If sequence is causal then limits of n

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

$$X(Z) = x(0) Z^0 + x(1) Z^{-1} + x(2) Z^{-2} + \dots + x(n) Z^{-n}$$

$$x(0) = a_0, \quad x(1) = a_1, \quad \dots \quad x(n) = a_n$$

$$x(n) = a_n \quad n \geq 0$$

Rules of Power Series Expansion $\rightarrow$

In case of anti causal sequence carryout the long division by writing the polynomials in the reverse order that means starting with the most negative term on left

$$|Z| < 1$$

In case of Causal sequence, carryout the long division without changing order of the polynomial. That means starting with the most +ve term on the left.

using power series method for determining Inverse Z transform (IZT) of

$$X(Z) = \frac{1}{1 - 1.5Z^{-1} + 0.5Z^{-2}}$$

for ROC $|Z| > 1$

ROC $|Z| < 0.5$

$$\begin{array}{r}
 1 + 1.5Z^{-1} + 1.75Z^{-2} + 1.875Z^{-3} \\
 1 - 1.5Z^{-1} + 0.5Z^{-2} \overline{) 1} \\
 \underline{+} \phantom{1.5Z^{-1} - 0.5Z^{-2}} \\
 1.5Z^{-1} - 0.5Z^{-2} \\
 1.5Z^{-1} - 2.25Z^{-2} + 0.75Z^{-3} \\
 \underline{+} \phantom{1.75Z^{-2} - 0.75Z^{-3}} \\
 1.75Z^{-2} - 0.75Z^{-3} \\
 1.75Z^{-2} - 2.625Z^{-3} + 0.85Z^{-4} \\
 \underline{+} \phantom{1.875Z^{-3} - 0.85Z^{-4}} \\
 1.875Z^{-3} - 0.85Z^{-4} \\
 1.875Z^{-3} - 2.812Z^{-4} + 0.93Z^{-5} \\
 \underline{+} \phantom{1.962Z^{-4} - 0.93Z^{-5}} \\
 1.962Z^{-4} - 0.93Z^{-5}
 \end{array}$$

$$x(z) = 1 + 1.5Z^{-1} + 1.75Z^{-2} + 1.875Z^{-3} + 1.962Z^{-4}$$

$$x(z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

$$\{1, 1.5, 1.75, 1.875, 1.962\}$$

$$\lambda(z) =$$

$$\frac{1}{0.5z^{-2} - 1.5z^{-1} + 1}$$

$$\begin{array}{r} 0.5z^{-2} - 1.5z^{-1} + 1 \overline{) 2z^2 + 6z^3 + 14z^4 + 30z^5} \\ \underline{1 - 3z + 2z^2} \\ 3z - 2z^2 \\ \underline{-3z + 9z^2 + 6z^3} \\ 7z^2 - 6z^3 \\ \underline{-7z^2 + 21z^3 + 14z^4} \\ 15z^3 - 14z^4 \\ \underline{-15z^3 + 45z^4 + 30z^5} \\ 31z^4 - 30z^5 \end{array}$$

$$x(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5$$

$$x(z) = \sum_{n=-1}^{-\infty} x(n) z^{-n}$$

$$x(n) = \{30, 14, 6, 2, 0, 0\}$$

$$\# \quad x(z) = \frac{z}{z-1} \quad |z| > 1$$

$$\# \quad x(z) = \frac{z}{z-a} \quad \frac{z}{-a+z}$$

Partial Fraction Expansion (P.F.E) Method $\Rightarrow$

$$x(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

Steps 1. Check given fn in proper form or not

(i) The coefficient a_0 in eqⁿ should be equal to 1 if $a_0 \neq 1$ then adjust polynomial

(ii) $a_n \neq 0$ and $M \geq N$. the long division is carried out to make $M < N$.

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Step 2: Multiply numerator & denominator by Z^N , means convert the in terms of +ve power of Z .

③ obtain eqn $\frac{X(Z)}{Z}$.

④ Fraction the denominator & obtain the roots' $(Z-P_1)(Z-P_2) \dots (Z-P_N) \dots$ P_1, P_2, P_N called as poles

⑤ while down the eqn in partial fraction expansion form

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-P_1} + \frac{A_2}{Z-P_2} + \dots + \frac{A_N}{Z-P_N}$$

Here $A_1, A_2, \dots, A_N$. The Co-efficient A_K is calculated as

$$A_K = (Z-P_K) \frac{X(Z)}{Z} \Big|_{Z=P_K}$$

⑥ By Calculating values of $A_1, A_2, \dots, A_N$ transfer Z to R.H.S
Now standard Z transforms pairs to obtain inverse Z transform

$$a^n u(n) \longrightarrow \frac{Z}{Z-a} \quad |Z| > |a|$$

$$-a^n u(-n-1) \longrightarrow \frac{Z}{Z-a} \quad |Z| < |a|$$

$$x(n) = |ZT \quad \frac{Z}{Z-P_K} = (P_K)^n u(n) \quad |Z| > |P_K|$$

$$x(n) = |ZT \quad \frac{Z}{Z-P_K} = -(P_K)^n u(-n-1) \quad |Z| < |P_K|$$

Find inverse Z of the following

$$X(Z) = \frac{1 - 1/2 Z^{-1}}{1 + \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}} \quad |Z| > \frac{1}{2}$$

Step 1

Check whether given fn is in proper form

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$$q_0 = 1, M = 1, N = 2$$

$$M < N \text{ and } q_0 = 1$$

Step 2 $\Rightarrow$

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$X(z) = \frac{z^2 [1 - \frac{1}{2}z^{-1}]}{z^2 [1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}]}$$

$$X(z) = \frac{z^2 - \frac{1}{2}z}{z^2 + \frac{3}{4}z + \frac{1}{8}} \Rightarrow z \left[\frac{z - \frac{1}{2}}{z^2 + \frac{3}{4}z + \frac{1}{8}} \right]$$

Step 3 $\rightarrow$

$$\frac{X(z)}{z} = \frac{z - \frac{1}{2}}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

Step 4 $\rightarrow$

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z^2 + \frac{3}{4}z + \frac{1}{8} \quad a=1, b=\frac{3}{4}, c=\frac{1}{8}$$

$$P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - 4(1 \times \frac{1}{8})}}{2 \times 1} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}}{2}$$

$$P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9-8}{16}}}{2} = \frac{-\frac{3}{4} \pm \frac{1}{4}}{2} = -\frac{3}{8} \pm \frac{1}{8}$$

$$P_1 = -\frac{1}{2}$$

$$P_2 = -\frac{1}{4}$$

step 5

$$\frac{X(z)}{z} = \frac{z^{-\frac{1}{2}}}{(z-p_1)(z-p_2)} = \frac{z^{-\frac{1}{2}}}{(z+\frac{1}{2})(z+\frac{1}{4})}$$

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$$\frac{X(z)}{z} = \frac{A_1}{(z+\frac{1}{2})} + \frac{A_2}{(z+\frac{1}{4})}$$

step 6

obtain values of A_1 & A_2

$$A_k = (z-p_k) \frac{X(z)}{z} \Big|_{z=p_k}$$

$$A_1 = (z-p_1) \frac{X(z)}{z} \Big|_{z=p_1}$$

$$p_1 = -\frac{1}{2}$$

$$A_1 = (z+\frac{1}{2}) \frac{X(z)}{z} \Big|_{z=-\frac{1}{2}}$$

$$A_1 = \frac{z-\frac{1}{2}}{z+\frac{1}{4}} \Big|_{z=-\frac{1}{2}} = \frac{-\frac{1}{2}-\frac{1}{2}}{-\frac{1}{2}+\frac{1}{4}} = \frac{-1}{\frac{-2+1}{4}} = \frac{-1}{\frac{-1}{4}} = 4$$

$$A_2 = \frac{z-\frac{1}{2}}{z+\frac{1}{2}} \Big|_{z=-\frac{1}{4}} = \frac{-\frac{1}{4}-\frac{1}{2}}{-\frac{1}{4}+\frac{1}{2}} = \frac{-1-2}{-1+2} = \frac{-3}{1} = -3$$

$$A_1 = 4, \quad A_2 = -3$$

step 7

$$\rightarrow \frac{X(z)}{z} = \frac{4}{z+\frac{1}{2}} - \frac{3}{z+\frac{1}{4}}$$

$$X(z) = \frac{4z}{z+\frac{1}{2}} - \frac{3z}{z+\frac{1}{4}}$$

Step 8 The 1st term of eqn (7) is $4 \cdot \frac{z}{z + \frac{1}{2}}$. Causal when $|z| > \frac{1}{2}$ (29)

$$1ZT \left[\frac{z}{z - p_k} \right] = (p_k)^n u(n) \quad |z| > |p_k|$$

$$1ZT \frac{z}{z + \frac{1}{2}} = \left(-\frac{1}{2} \right)^n u(n)$$

$$1ZT \frac{z}{z + \frac{1}{4}} = \left(-\frac{1}{4} \right)^n u(n)$$

$$x(n) = 4 \left(-\frac{1}{2} \right)^n u(n) - 3 \left(-\frac{1}{4} \right)^n u(n)$$

Determine all possible sequences $x(n)$ associated with Z transform

$$X(z) = \frac{5z^{-1}}{(1 - 2z^{-1})(3 - z^{-1})}$$

$$X(z) = \frac{5z^{-1}}{3 - 7z^{-1} - 2z^{-2}}$$

$M=1, N=2 \quad M < N \quad a_0=3$ we want $a_0=1$

$$\frac{\frac{5}{3}z^{-1}}{1 - \frac{7}{3}z^{-1} - \frac{2}{3}z^{-2}}$$

Step 2

$$X(z) = \frac{\frac{5}{3}z}{z^2 - \frac{7}{3}z - \frac{2}{3}}$$

Step 3

$$\frac{X(z)}{z} = \frac{\frac{5}{3}}{z^2 - \frac{7}{3}z - \frac{2}{3}}$$

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step 4

$$p_1 = 2 \quad p_2 = \frac{1}{3}$$

step 5

$$\frac{X(z)}{z} = \frac{\frac{5}{3}}{(z-2)(z-\frac{1}{3})}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-2} + \frac{A_2}{z-\frac{1}{3}}$$

step 6

$$A_1 = \frac{\frac{5}{3}}{(z-\frac{1}{3})} \Big|_{z=2} = \frac{\frac{5}{3}}{2-\frac{1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$$

$$A_2 = \frac{\frac{5}{3}}{(z-2)} \Big|_{z=\frac{1}{3}} = \frac{\frac{5}{3}}{\frac{1}{3}-2} = \frac{\frac{5}{3}}{-\frac{5}{3}} = -1$$

step 7

$$\frac{X(z)}{z} = \frac{1}{z-2} - \frac{1}{z-\frac{1}{3}}$$

$$X(z) = \frac{z}{z-2} - \frac{z}{z-\frac{1}{3}}$$

(i) when ROC

$$|z| > 2 \quad = 2^n u(n) - \left(\frac{1}{3}\right)^n u(n)$$

(ii) when ROC is $2 > |z| > \frac{1}{3}$

$$- (2)^n u(-n-1) + \left(\frac{1}{3}\right)^n u(n)$$

(iii) when ROC $|z| < \frac{1}{3}$

$$- (2)^n u(-n-1) - \left(\frac{1}{3}\right)^n u(-n-1)$$

7) Determine IZT of

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$$X(z) = \frac{4z^2 + 3z^{-1} + 2}{\frac{1}{8}z^2 - \frac{3}{8}z^{-1} + 1} \quad \text{for Causal sequence}$$

$$a_0 = 1, M=2, N=2$$

$$\frac{\frac{1}{8}z^2 - \frac{3}{8}z^{-1} + 1}{\frac{8}{4z^2 + 3z^{-1} + 2}} = \frac{4z^2 - 3z^{-1} + 8}{4z^2 + 3z^{-1} + 2}$$

$$X(z) = 8 + \frac{15z^{-1} - 6}{\frac{1}{8}z^2 - \frac{3}{8}z^{-1} + 1}$$

$$X_1(z) = \frac{15z^{-1} - 6}{\frac{1}{8}z^2 - \frac{3}{8}z^{-1} + 1} \quad M=1, N=2$$

$$\frac{15z - 6z^2}{\frac{1}{8} - \frac{3}{8}z + z^2} = \frac{-6z^2 + 15z}{z^2 - \frac{3}{8}z + \frac{1}{8}}$$

$$\frac{X(z)}{z} = \frac{-6z + 15}{z^2 - \frac{3}{8}z + \frac{1}{8}}$$

Find inverse z transform using convolution method

$$X(z) = \frac{z^2}{\left(z - \frac{1}{4}\right)^2} \cdot \text{ROC } |z| > \frac{1}{4}$$

According to Conv property

$$x_1(n) * x_2(n) \Leftrightarrow X_1(z) \cdot X_2(z)$$

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$$\text{Let } x_1(z) = \frac{z}{z - \frac{1}{4}} \quad x_2(z) = \frac{z}{z - \frac{1}{4}}$$

$$x(z) = x_1(z) \cdot x_2(z)$$

We have standard Z Transform formula

$$z^n u(n) \xrightarrow{z} \frac{z}{z - a} \quad |z| > |a|$$

$$x_1(n) = \left(\frac{1}{4}\right)^n u(n) \quad x_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

$$x(n) = x_1(n) * x_2(n)$$

According to definition of Convolution

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$x_1(k) \cdot \left(\frac{1}{4}\right)^k u(k) \quad x_2(n-k) = \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

$$x(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k u(k) \cdot \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k \cdot \left(\frac{1}{4}\right)^n \cdot \left(\frac{1}{4}\right)^{-k} u(k) \cdot u(-k+n)$$

$$x(n) = \left(\frac{1}{4}\right)^n \sum_{k=-\infty}^{\infty} u(k) u(-k+n)$$

$$\left(\frac{1}{4}\right)^n \sum_{k=0}^n 1$$



$$\sum_{k=0}^n 1 = \sum_{k=0}^n 1 \cdot 1 = \sum_{k=0}^n 1 = n+1$$

$$\rightarrow x(n) = \left(\frac{1}{4}\right)^n \cdot \sum_{k=0}^n 1 = \left(\frac{1}{4}\right)^n (n+1)$$

$$\rightarrow x(n) = \left(\frac{1}{4}\right)^n \cdot (n+1) u(n)$$

$$* X(z) = \frac{1}{1 - 3z^{-1} + 2z^{-2}}$$

$$X(z) = \frac{z^2}{z^2 - 3z + 2} = \frac{z^2}{(z-1)(z-2)}$$

$$X_1(z) = \frac{z}{z-1} \quad X_2(z) = \frac{z}{z-2}$$

$$x_1(n) * x_2(n) \longrightarrow X_1(z) \cdot X_2(z)$$

$$x_1(n) = u(n) \quad x_2(n) = 2^n u(n)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$\sum_{k=-\infty}^{\infty} u(k) 2^{n-k} u(n-k)$$

$$\sum_{k=-\infty}^{\infty} u(k) 2^n \cdot 2^{-k} u(n-k)$$

$$2^n \sum_{k=-\infty}^{\infty} 2^{-k} u(k) u(n-k)$$

$$2^n \sum_{k=0}^n \left(\frac{1}{2}\right)^k$$

$$\sum_{k=N_1}^{N_2} A^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$2^n \left[\frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{n+1}}{1 - 1/2} \right] = 2^n \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{2-1}{2}} \right]$$

$$2^n \cdot 2 \left[1 - \left(\frac{1}{2}\right)^{n+1} \right] \Rightarrow 2^{n+1} \left[1 - \left(\frac{1}{2}\right)^{n+1} \right] u(n) \underline{\underline{Ans}}$$

$$x(z) = \frac{1}{(z-1)(z-3)}$$

121 using residue Method

$$x(z) \cdot z^{n-1} = \frac{z^{n-1}}{(z-1)(z-3)}$$

Poles $z=1, z=3$

$$x(z) \cdot z^{n-1} = \frac{(1)^n \cdot (1)^{-1}}{(1-3)} = \frac{(1)^n \cdot 1}{-2} = 0.5 (1)^n$$

$$\frac{(3)^n \cdot (3)^{-1}}{3-1} = \frac{(3)^n \cdot \frac{1}{3}}{2} = \frac{(3)^n \cdot 0.33}{2} = 0.17 (3)^n$$

$$x(n) = 0.5 (1)^n + 0.17 (3)^n \quad \text{Ans}$$

Find $x(n)$ for given $X(Z)$ using Residue Method (34)

$$X(Z) = \frac{10Z}{(Z-1)(Z-2)} \quad \text{--- (1)}$$

Step 1 Multiply both sides of eq (1) by Z^{n-1} .

$$X(Z) \cdot Z^{n-1} = \frac{10Z^1 \times Z^{n-1}}{(Z-1)(Z-2)} = \frac{10Z^n}{(Z-1)(Z-2)}$$

Step 2 Poles $P=1, P=2$

Thus residue at $Z=1$ is

$$R_{Z=1} = \left. \frac{(Z-1) \cdot 10Z^n}{Z-2} \right|_{Z=1} = \frac{10 \cdot (1)^n}{1-2}$$

$$R_{Z=1} = -10(1)^n$$

Residue at $Z=2$ is

$$R_{Z=2} = \left. \frac{(Z-2) \cdot 10Z^n}{Z-1} \right|_{Z=2} = \frac{10(2)^n}{2-1}$$

$$R_{Z=2} = 10(2)^n$$

$$x(n) = -10(1)^n + 10(2)^n \quad \text{for } n \geq 0$$

Relationship b/w DFT & ZT $\Rightarrow$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

We know that $Z = e^{j\omega}$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Now Suppose $x(z)$ is sampled at 'N' equally spaced pts on the unit circle. (35)

$$\omega = \frac{2\pi k}{N}$$

$$z = e^{j\frac{2\pi k}{N}}$$

$$x(z) = \sum_{n=0}^{\infty} x(n) e^{-j\frac{2\pi k n}{N}} \text{ at } z = e^{j\frac{2\pi k}{N}}$$

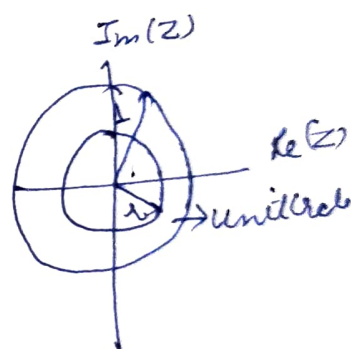
if causal sequence and has N no. of samples then we can write it as

$$x(k) = x(z) \rightarrow x(z) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k n}{N}}$$

Condition for causality and stability in terms of z transform.

① For a causal system we know that ROC is exterior part of circle ^{that means} $|z| > r$

② For stable ckt ROC must include unit circle. having radius 1
 $r < 1$



Combined condition for causal + stable system

$$|z| > r < 1$$

A causal LTI system has transfer fn

$$H(z) = \frac{(1 - 0.5z^{-1})(1 - z^{-1})}{(1 + 0.2z^{-1})(1 + 0.8z^{-1})(1 - 0.8z^{-1})}$$

- (i) Give ROC Condition
- (ii) Show pole zero diag of system
- (iii) Find impulse response of system
- (iv) Comment on system stability

(26)

$$H(z) = \frac{(1 - 0.5z^{-1})(1 - z^{-1})}{(1 + 0.2z^{-1})(1 + 0.8z^{-1})(1 - 0.8z^{-1})}$$

To convert into +ve power of z multiply N/D by z^3

$$\frac{z \cdot z(1 - 0.5z^{-1}) \cdot z(1 - z^{-1})}{z(1 + 0.2z^{-1})z(1 + 0.8z^{-1})z(1 - 0.8z^{-1})} \quad \text{--- (1)}$$

$$\frac{z \cdot (z - 0.5)(z - 1)}{(z + 0.2)(z + 0.8)(z - 0.8)} \quad \text{--- (2)}$$

Poles at $-0.2, -0.8, 0.8$

Requires $\text{ROC } |z| > 0.8$

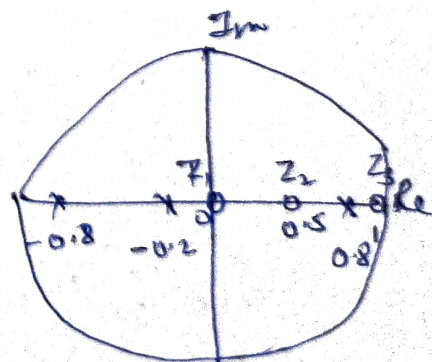
The given system is causal

(ii)

$$H(z) = \frac{(z - z_1)(z - z_2)(z - z_3)}{(z - p_1)(z - p_2)(z - p_3)} \quad \text{--- (3)}$$

$$z_1 = 0, \quad z_2 = 0.5, \quad z_3 = 1$$

$$p_1 = -0.2, \quad p_2 = -0.8, \quad p_3 = +0.8$$



(iii)

$$\frac{H(z)}{z} = \frac{(z - 0.5)(z - 1)}{(z + 0.2)(z + 0.8)(z - 0.8)}$$

$$\frac{H(z)}{z} = \frac{A_1}{z + 0.2} + \frac{A_2}{z + 0.8} + \frac{A_3}{z - 0.8}$$

$$A_1 = \frac{(-0.2 - 0.5)(-0.2 - 1)}{((-0.2 + 0.8))(-0.2 - 0.8)} \bigg|_{z = -0.2}$$

$$= \frac{(-0.7)(-1.2)}{(0.6)(-1)} = -1.4$$

$$A_2 = \frac{(z - 0.5)(z - 1)}{(z + 0.2)(z + 0.8)} \bigg|_{z = -0.8}$$

$$\frac{(-0.8 - 0.5)(-0.8 - 1)}{(-0.8 - 0.2)(-0.8 - 0.8)} = 2.44$$

$$A_3 = \frac{(z - 0.5)(z - 1)}{(z + 0.2)(z + 0.8)} \bigg|_{z = 0.8}$$

$$\frac{(0.8 - 0.5)(0.8 - 1)}{(0.8 + 0.2)(0.8 + 0.8)} = -0.0375$$

$$\frac{H(z)}{z} = \frac{-1.4}{z + 0.2} + \frac{2.44}{z + 0.8} - \frac{0.0375}{z - 0.8}$$

$$H(z) = \frac{-1.4z}{z + 0.2} + \frac{2.44z}{z + 0.8} - \frac{0.0375z}{z - 0.8}$$

$$h(n) = -1.4(0.2)^n u(n) + 2.44(-0.8)^n u(n) - 0.0375(0.8)^n u(n)$$

All the poles lie inside the unit circle so system is stable